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1.1 - Solving Systems of Linear Equations

Define: a *linear equation* in variables $x_1, x_2, \ldots, x_n$
and numbers $a_1, a_2, \ldots, a_n$ and b
is an equation

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n = b$$

e.g.

Define: a *system* of linear equations is a collection of linear equations

e.g.

Define: $(s_1, \ldots, s_n)$ is a *solution* for a system of equations
if plugging in $s_1, \ldots, s_n$ for $x_1, \ldots, x_n$ makes *ALL* the equations in the system true.

e.g.

Define: The *solution set* of a system is the *SET of ALL solutions*.

If $n = 2$, then solutions are points in the plane
and *solution sets are graphs* in 2D

If $n = 3$, then solutions are points in 3D
and *solution sets are graphs* in 3D

NOTE: "Solve" means "find the solution set".

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Row Operations

1. **Scaling** Multiply row by nonzero number

   E.g. Set $r_2^* = \frac{1}{2} r_2$

   Intended use: Turn the first nonzero entry in r_2 into a 1.

2. **Replacement** Add multiple of one row to another row

   E.g. Set $r_2^* = r_2 + 4 \cdot r_1$

   Intended use: Use the first nonzero entry in r_1 to zero out an entry in r_2.

   Note: the only row that changes is r_2 (the row being replaced).

3. **Interchange:** Switch two rows.

   E.g. Set $r_2^* = r_3$ and $r_3^* = r_2$.

   Intended use: Rearrange the row's so that the matrix is more beautiful.

The above row operations **DO NOT** add or remove solutions from the original system *because* they correspond to procedures which generate equivalent equations

1. Scaling: Multiply (or divide) both sides of the equation by the same nonzero real number.
2. Replacement: Add (or subtract) the same real number to (from) both sides of the equation.
3. Interchange: Write the same equations down in a different order.

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1.2 - Echelon Form, Reduced Echelon Form, and Pivots

Define: A matrix is in *echelon form* (aka row echelon form) if
its nonzero leading entries move down and to the right, never skipping a row

Define: An echelon form matrix is in *reduced echelon form*
if it also satisfies

1. all leading nonzero row entries are 1
2. there are zero's *above* each leading non-zero entry

Note: Matrices have *many* equivalent echelon forms, **BUT**

Theorem 1: Each matrix is equivalent to a **unique** *reduced* echelon matrix.

Useful Notation: *pivot positions*

Note: Reducing an echelon form matrix does *not* change the position of its leading nonzero terms.

Define: Fix a matrix A.

- a *pivot position* in A is a location in A that correspons to a leading 1 in the reduced echelon form of A, and
- a *pivot column* is a column of A that contains a pivot position.

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The Row-Reduction Algorithm

A recipe that will row-reduce *any* matrix. The algorithm has two "phases."

Forward Phase: Reduce to echelon form FAST

Note: This phase moves from LEFT TO RIGHT

1. (a) Start with the *leftmost non-zero column.* This is a pivot column.
   (b) Ensure the top position is a pivot by interchanging rows (if necessary).

2. Use Replacement to create zero's in all positions *below* the pivot.

3. Repeat steps 1-3 for the circled submatrix. Keep going until the matrix is in echelon form.

Backward Phase: Clearing out the pivot columns.

Note: This phase moves from RIGHT TO LEFT

1. Starting with *rightmost pivot*, and working up and to the left, create zeros above each pivot. If the pivot is not 1, rescale the row to make it 1.

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1.3 - Vectors, Linear Combinations, and Vector Equations

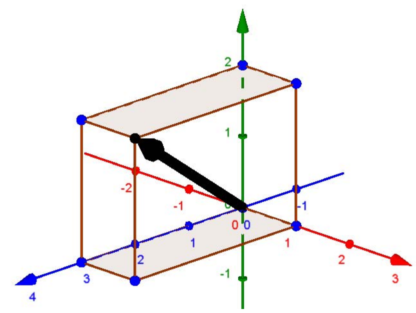
Define: a vector $\vec{\mathbf{v}}$ in $\mathbb{R}^m$ is an $m \times 1$ matrix

$$\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

where $v_1, v_2, \ldots, v_m$ are scalar numbers in $\mathbb{R}$.

A vector in $\mathbb{R}^3$ is a vector in 3D space.

E.g. Graph the vector $\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$



We can *add* and *scale* vectors as follows.
If $c \in \mathbb{R}$ and $\vec{\mathbf{v}} \in \mathbb{R}^m$, then you scale each component

$$c \cdot \vec{\mathbf{v}} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_m \end{bmatrix}$$

If $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^m$, then you add component-wise

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} + \begin{bmatrix} 2_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_m + w_m \end{bmatrix}$$

We write $\vec{\mathbf{0}}$ for the all-zero vector. That is, $\vec{\mathbf{0}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Algebraic Properties of $\mathbb{R}^m$
For all $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ and scalars c and d,

- $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$
- $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}} = \vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}})$
- $\vec{\mathbf{u}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{u}} = \vec{\mathbf{u}}$
- $\vec{\mathbf{u}} + (-\vec{\mathbf{u}}) = \vec{\mathbf{0}}$
- $c \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}}) = c \cdot \vec{\mathbf{u}} + c \cdot \vec{\mathbf{v}}$
- $(c + d) \cdot \vec{\mathbf{u}} = c \cdot \vec{\mathbf{u}} + d \cdot \vec{\mathbf{u}}$
- $c(d\vec{\mathbf{u}}) = (cd)\vec{\mathbf{u}}$
- $1 \cdot \vec{\mathbf{u}} = \vec{\mathbf{u}}$

Proof of commutativity: $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} w_1 + v_1 \\ w_2 + v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{\mathbf{w}} + \vec{\mathbf{v}}$

NOTICE: We don't define vector multiplication. We want to study **linear** combinations of vectors.

Define: Give vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n, \vec{\mathbf{b}}$ in $\mathbb{R}^m$ (with m components each)
and numbers $c_1, c_2, \ldots, c_n$ in $\mathbb{R}$

If $\vec{\mathbf{b}} = c_1 \cdot \vec{\mathbf{a}}_1 + c_2 \cdot \vec{\mathbf{a}}_2 + \cdots + c_n \cdot \vec{\mathbf{a}}_n$
Then $\vec{\mathbf{b}}$ is a *linear combination* of $\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n$

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The Span of a Set of Vectors

Define: The subset of $\mathbb{R}^m$ *spanned by the vectors* $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n$ in $\mathbb{R}^m$ is the set of all linear combinations of $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots, \vec{\mathbf{a}}_n$.

Formally:
$\text{Span}\{\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n\} = \left\{\vec{\mathbf{b}} \in \mathbb{R}^m : \vec{\mathbf{b}} = c_1 \cdot \vec{\mathbf{a}}_1 + \cdots + c_n \cdot \vec{\mathbf{a}}_m \text{ for some scalar numbers } c_1, \dots, c_n \in \mathbb{R}\right\}$

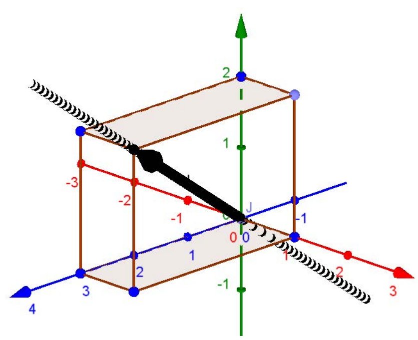
Intuitively:
$\text{Span}\{\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n\}$ is the set of vectors $\vec{\mathbf{b}}$ that can be "gotten" from linear combinations of $\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n$.

Geometric Description of Span

If $\vec{\mathbf{v}}, \vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, then

1. $\text{Span}\{\vec{\mathbf{v}}\}$ is a line through $\vec{\mathbf{v}}$ and $\vec{\mathbf{0}}$.

   E.g. Let $\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Graph $\text{Span}\{\vec{\mathbf{v}}\}$.

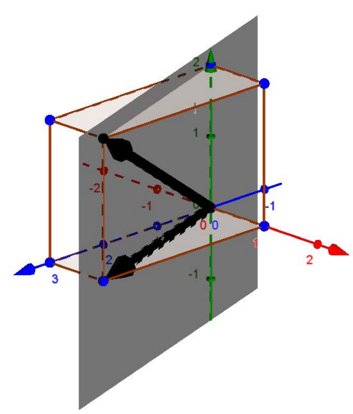


2. $\text{Span}\{\frac{1}{3}\vec{\mathbf{v}}, \frac{3}{4}\vec{\mathbf{v}}\}$ is the same line through $\vec{\mathbf{v}}$ and $\vec{\mathbf{0}}$.

3. If $\vec{\mathbf{w}}$ is *not* a multiple of $\vec{\mathbf{v}}$ then $\text{Span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ is a *plane* through the points $\vec{\mathbf{v}}, \vec{\mathbf{w}}, \vec{\mathbf{0}}$.

   E.g. Suppose $\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

   Graph $\text{Span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$.



NOTICE: *Every* plane through $\vec{\mathbf{0}}$ is the span of two non-zero vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$

Saved for future classes: *i*. planes *not* going through $\vec{\mathbf{0}}$ (§1.5), and *ii*. the span of > 2 vectors (§1.7)

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Summary: Choose the Notation Best for your Purposes

	Systems of Equations	Augmented Matrices	Vector Equations	Matrix Equations
Algebraic meaning	√		√	? (!)
Geometric meaning	√		√	? (!)
Concise to Write		√++	√	√+
Easy to Work with		√++		√
Explicitly States Variables	√		√	√

For now, vector equations seem like the clear winner for "clarity of meaning" and augmented matrices are the clear winner for "ease of use".

As we learn more about Matrices (§1.9 and onward) they will pull ahead in "clarity of meaning".

Four ways of saying the same thing

System of Equations

$$\begin{aligned} 2x + 3y - 2z &= 6 \\ 5x + 2y + z &= 10 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & 6 \\ 5 & 2 & 1 & 10 \end{array}\right]$$

Vector Equation

$$x \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + y \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} + z \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Matrix Equation

$$\begin{bmatrix} 2 & 3 & -2 \\ 5 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Theorem 3: If an $m \times n$ matrix A has columns $\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n$, then the following have the same solution set:

1. $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$
2. $x_1 \cdot \vec{\mathbf{a}}_1 + x_2 \cdot \vec{\mathbf{a}}_2 + \cdots + x_n \cdot \vec{\mathbf{a}}_n = \vec{\mathbf{b}}$
3. the linear system with augmented matrix $\left[\begin{array}{cccc|c} \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \ldots & \vec{\mathbf{a}}_n & \vec{\mathbf{b}} \end{array}\right]$

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$A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution

$\iff$ $[A|\vec{\mathbf{b}}]$ is consistent

$\iff$ reduced echelon form AVOIDS $[0\dots0|\blacksquare]$.

Eg:
$\left[\begin{array}{ccc|c} 0 & \blacksquare & 0 & b_1 \\ 0 & 0 & \blacksquare & b_2 \end{array}\right]$ is consistent for *every* $\vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ because there is *a pivot* in each **row**

Eg:
$\left[\begin{array}{ccc|c} \blacksquare & \star & 0 & b_1 \\ 0 & 0 & 0 & b_2 \end{array}\right]$ is *inconsistent* for *some* $\vec{\mathbf{b}}$ because there is *no* pivot in second **row**

Notice:
The reduced echelon form of $[A|\vec{\mathbf{b}}]$ *never contains* the row $[0\dots0|\blacksquare]$, *regardless* of the vector $\vec{\mathbf{b}}$

$\iff$

The reduced echelon form of $[A]$ (the reduced coefficient matrix *only*) has a pivot in each **row**.

Putting this together with Theorem 3, we get:

Theorem 4: If A is an $m \times n$ matrix, *the following are equivalent*:

1. A$\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for *every* $\vec{\mathbf{b}}$ in $\mathbb{R}^m$
2. each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is a linear combination of columns of A
3. the columns of A Span $\mathbb{R}^m$
4. A has a pivot in each **row**.